

ON CYCLES IN GRAPHS WITH SPECIFIED RADIUS AND DIAMETER

PAVEL HRNČIAR

ABSTRACT. Let G be a graph of radius r and diameter d with $d \leq 2r - 2$. We show that G contains a cycle of length at least $4r - 2d$, i.e. for its circumference it holds $c(G) \geq 4r - 2d$. Moreover, for all positive integers r and d with $r \leq d \leq 2r - 2$ there exists a graph of radius r and diameter d with circumference $4r - 2d$.

For a connected graph G , the *distance* $d_G(u, v)$ or briefly $d(u, v)$ between a pair of vertices u and v is the length of a shortest path joining them. The distance between a vertex $u \in V(G)$ and a subgraph H of G will be denoted by $d(u, H)$, i.e. $d(u, H) = \min\{d(u, v); v \in V(H)\}$. The *eccentricity* $e_G(u)$ (briefly $e(u)$) of a vertex u of G is the distance of u to a vertex farthest from u in G , i.e. $e_G(u) = \max\{d_G(u, v); v \in V(G)\}$. The *radius* $\text{rad } G$ of G is the minimum eccentricity among the vertices of G while the *diameter* $\text{diam } G$ of G is the maximum eccentricity. The *circumference* of a graph G , denoted $c(G)$, is the length of any longest cycle in G .

A path P (a cycle C) in G is called *geodesic* if for any two vertices of P (of C) their distance in P (in C) equals their distance in G . A nontrivial connected graph with no cut-vertices is called a *nonseparable graph*. A *block* of a graph G is a maximal nonseparable subgraph of G .

A connected unicyclic graph G with the cycle C is called a *sun-graph* (see [2]) if $\deg_G(u) \leq 3$ for $u \in V(C)$ and $\deg_G(u) \leq 2$ for $u \in V(G) \setminus V(C)$. A $u - v$ path P in a sun-graph G is called a *ray* if $V(P) \cap V(C) = \{u\}$ and $\deg_G(v) = 1$. A sun-graph with the cycle C_m of length m and with m rays of length k will be denoted by $S_{m,k}$.

In what follows we answer a question that was posed several decades ago in [3]: "How large a cycle must there be in a graph of radius m and diameter n ? This question is also open. For radius 3 and diameter 4, the graph must have a cycle of length at least 4, which can be verified by brute force techniques The situation in general is unclear."

Our main result is the following theorem.

Theorem 1. *Let G be a graph of radius r and diameter d with $d \leq 2r - 2$. Then $c(G) \geq 4r - 2d$.*

Proof. Since $d \leq 2r - 2$, G is not a tree. Let C be a cycle of G and B be a block of G containing C . Suppose, contrary to our claim, that $c(G) < 4r - 2d$. Since B is a nonseparable subgraph of G , every two vertices of B lie on a common cycle (see [1, Theorem 1.6]). We get $\text{diam } B \leq 2r - d - 1 \leq r - 1$ (and so $B \neq G$).

Let u be a vertex such that $d(u, B) = \max\{d(v, B); v \in V(G)\}$ and let $u_B \in V(B)$ be a vertex with $d(u, u_B) = d(u, B)$. Evidently, u_B is a cut-vertex of G .

Let G_1 be a component of $G - u_B$ containing the vertex u . Put $d(u, u_B) = a$. We distinguish two cases.

(1) $a \leq r - 1$

Let v be a vertex of G . If $v \in V(G_1)$ then $d(v, u_B) \leq a \leq r - 1$. If $v \in V(B)$ then $d(v, u_B) \leq \text{diam } B \leq r - 1$. Let, finally, $v \in V(G) \setminus (V(G_1) \cup V(B))$ and $v_B \in V(B)$ be a vertex such that $d(v, v_B) = d(v, B)$. Evidently, v_B is a cut-vertex of G and $d_G(u_B, v_B) = d_B(u_B, v_B)$. Denote $d(v, v_B) = b$ and $d(u_B, v_B) = c$. Suppose first that $b + c \geq r$. We have $c \leq \text{diam } B \leq 2r - d - 1$

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and $b \leq a$. Then $d(u, v) = d(u, u_B) + d(u_B, v_B) + d(v_B, v) = a + c + b \geq 2b + c \geq 2(r - c) + c \geq 2r - (2r - d - 1) = d + 1$. Since $\text{diam } G = d$, we get $b + c \leq r - 1$ and so $d(u_B, v) \leq r - 1$. Finally, we have $e(u_B) \leq r - 1$, a contradiction.

(2) $a \geq r$

Let u_1 be a vertex of a geodesic $u - u_B$ path P^1 with $d(u, u_1) = r - 1$. If w is a vertex from $V(G) \setminus V(G_1)$ then u_B is on a geodesic $w - u_1$ path and we get $d(w, u_1) \leq r - 1$ (since $d(w, u) \leq 2r - 2$). Since $e(u_1) \geq r$ (otherwise we have a contradiction), there is a vertex $v \in V(G_1)$ such that $d_{G_1}(v, u_1) = d_G(v, u_1) \geq r$. Let P^2 be a geodesic $v - u_B$ path and let v_1 be the first vertex of P^2 which is on P^1 . Since $d(v, u_B) \leq d(u, u_B)$, we get $d(u_B, v_1) < d(u_B, u_1)$. Let P^3 be a geodesic $v - u$ path and let v_2 be the first of its vertices which is on P^1 . It is obvious (since $d(v, u) \leq 2r - 2$) that $d(u_B, v_2) > d(u_B, u_1)$. Evidently, there is a cycle C' such that $\{v_1, v_2\} \subseteq V(C')$.

Let G_2 be a subgraph of G induced by the set $V(G_1) \cup \{u_B\}$. Let $w \in V(G) \setminus V(G_2)$ be such a vertex that $d(w, u_B) = \max\{d(x, u_B); x \in V(G) \setminus V(G_2)\}$ and P be a geodesic $w - u_B$ path. Consider a graph G' for which $V(G') = V(G_2) \cup V(P)$ and $E(G') = E(G_2) \cup E(P)$. It is obvious that $|V(G')| < |V(G)|$ and if there is a vertex $z \in V(G')$ with $e_{G'}(z) \leq r - 1$ then $e_G(z) \leq r - 1$, too.

We can repeat the previous considerations with the graph G' and its block B' containing the cycle C' . It is clear now that after a finite number of the described steps we get a contradiction. \square

Corollary 2. *If G is a graph with $\text{rad } G = r$ and $\text{diam } G \leq 2r - 2$, then G contains a cycle of length at least 4, i.e. $c(G) \geq 4$.*

Corollary 3. *If $c(G) = 3$ and $\text{rad } G = r$, then $\text{diam } G \in \{2r - 1, 2r\}$.*

For all positive integers r and d satisfying $r \leq d \leq 2r - 2$ there exists an infinite number of graphs of radius r , diameter d and circumference $4r - 2d$. One of these graphs is C_{2r} for $d = r$. If $d > r$, one of these graphs is $S_{4r-2d, d-r}$, i.e. a sun-graph with the cycle C_{4r-2d} and with $4r - 2d$ rays of length $d - r$ (see Figure 1a for $r = 3, d = 4$ and Figure 1b for $r = 5, d = 7$).

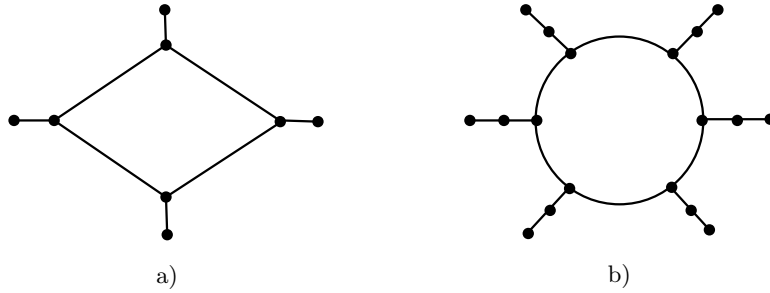


FIGURE 1.

Now it is a simple matter to find infinite classes of graphs with mentioned properties (see Figure 2 for an inspiration).

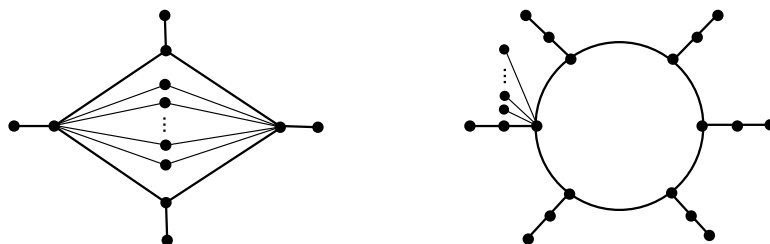


FIGURE 2.

It is known that if a graph G with radius r and diameter $d \leq 2r - 2$ has at most $3r - 2$ vertices, then it holds $c(G) \geq 2r$. This fact is a consequence of the following theorem (see [2]).

Theorem 4. [2] *Let G be a graph with $\text{rad } G = r$, $\text{diam } G \leq 2r - 2$, on at most $3r - 2$ vertices. Then G contains a geodesic cycle of length $2r$ or $2r + 1$.*

Using Theorem 4 it is easy to find all nonisomorphic graphs of minimal order and specified radius and diameter (see [3],[2]).

Let G be a sun-graph with the cycle C_{2r-1} ($r \geq 3$), with r rays of length 1 and such that exactly two of its end-vertices have distance 3 (see Figure 3 for $r = 5$). It is easy to see that $\text{rad } G = r$, $|V(G)| = 3r - 1$ and $c(G) = 2r - 1$.

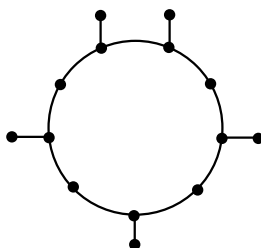


FIGURE 3.

We can conclude that the bound $3r - 2$ in Theorem 4 is the best possible (for $r \geq 3$).

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DEPARTMENT OF MATHEMATICS, FACULTY OF NATURAL SCIENCES, MATEJ BEL UNIVERSITY, TAJOVSKÉHO 40, 974 01 BANSKÁ BYSTRICA, SLOVAKIA

E-mail address: Pavel.Hrnciar@umb.sk